ONE-SIDED COMPLETE STABLE MINIMAL SURFACES

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Abstract

We prove that there are no complete one-sided stable minimal surfaces in the Euclidean 3-space. We classify least area surfaces in the quotient of $\mathbb{R}^3$ by one or two linearly independent translations and we give sharp upper bounds of the genus of compact two-sided index one minimal surfaces in non-negatively curved ambient spaces. Finally we estimate from below the index of complete minimal surfaces in flat spaces in terms of the topology of the surface.

1. Introduction

Stable minimal surfaces in 3-manifolds minimize area up to second order for any compactly supported smooth deformation. They play a fundamental role in minimal surface theory. Do Carmo and Peng [**6**], Fischer-Colbrie and Schoen [**11**] and Pogorelov [**26**], independently, proved that complete stable two-sided minimal surfaces immersed in the euclidean 3-space are flat. The results of [**11**] apply to 3-manifolds with nonnegative scalar curvature. In these ambient spaces, Fischer-Colbrie [**10**] proved that two-sided complete stable minimal surfaces with compact boundary have finite finite total curvature. Recently Meeks and Rosenberg [**19**] classified two-sided stable complete minimal surfaces in 3-manifolds of the type $M \times \mathbb{R}$, M being a closed Riemannian surface.

In the one-sided case the stability condition (and the area minimization problem in general) becomes more subtle. This case should not be considered as a minor or secondary one. On the contrary, it presents a rich and interesting geometry, even in flat or positively curved 3-manifolds, which is far from being understood. For instance, while two-sided stable complete minimal surfaces in flat 3-manifolds must be planar, Ross [**33**] has proved that the classical P and D Schwarz minimal surfaces are stable when considered as nonorientable surfaces with total curvature -4π in suitable 3-tori. As another example we point out

that it is a trivial fact that there are no two-sided compact stable minimal surfaces in positively curved 3-manifolds. However, for any positive integer n, there are lens spaces (metric quotients of the round 3-sphere) which admit one-sided closed surfaces with Euler characteristic smaller than $-n$ minimizing area in its isotopy class, see [**35**].

In this paper we study one-sided complete stable minimal surfaces (and related ones) in flat (or nonnegatively curved) 3-manifolds. When the ambient space is euclidean, we will prove that

> *there are no nonorientable complete stable minimal surfaces in $\mathbb{R}^3$.*

This theorem was previously known only in the finite genus case, Ross [**34**] (see also [**15**]). As an important consequence, we extend to one-sided surfaces the Schoen's curvature estimate for stable minimal surfaces in 3-manifolds, [**37**].

We also consider quotients $\mathbb{R}^3/\Gamma$ of Euclidean space, where Γ is a discrete group of translations of $\mathbb{R}^3$. We prove that *if* $\operatorname{rank}(\Gamma) = 1$ (*resp.* $\operatorname{rank}(\Gamma) = 2$), *then the only complete stable minimal surfaces in* $\mathbb{R}^3/\Gamma$ *are planar surfaces and the Helicoid* (*resp. Scherk doubly periodic surfaces*) *with total curvature* -2π. This result was proved by Ross and Schoen [**36**] assuming the surface has finite genus, see also [**3**, **27**]. The topology of the Helicoid and the Scherk surfaces above is the one of the projective plane once or twice punctured, respectively. In the embedded case, the second surface exists only when the planar lattice Γ is rectangular.

An important particular case of stable surfaces are area minimizing surfaces in various natural classes of surfaces (integer homology, $\mathbb{Z}_2$-homology, isotopy, homotopy, etc). It is well-known that the plane is the only complete area minimizing surface in $\mathbb{R}^3$. The classification of least area surfaces in general flat 3-manifolds is an interesting open problem in classical minimal surface theory. The two-sided case gives necessarily planar surfaces, but the list of one-sided area minimizing surfaces in flat 3-manifolds is not known at the present. The result below contains the first progress on this problem.

> *The Helicoid and the Scherk surfaces above are the only properly embedded nonflat area minimizing surfaces* mod 2 *in the quotients of* $\mathbb{R}^3$ *by one or two linearly independent translations, respectively.*

When the rank of Γ is 3 we are able to determine exactly the topology of stable surfaces. We prove that *closed stable minimal surfaces in flat 3-tori are either planar 2-tori or Klein bottles with a handle.* Planar 2-tori are all stable. In fact they are (the unique) area minimizing surfaces in their integer homology class. Klein bottles with a handle are one-sided, their total curvature is -4π and their orientable two-sheeted covering

is a genus 3 surface. Meeks has parameterized these surfaces as a 5-dimensional family, see [**17**]. The moduli spaces of stable and unstable minimal Klein bottles with a handle are, both, noncompact, see [**34**]. We do not have, at the moment, any bound on the topology of one-sided stable minimal surfaces in flat 3-manifolds obtained by discrete groups other than translations groups.

Besides stable minimal surfaces, minimal surfaces with index one have received a large amount of attention, see the examples constructed in [**4**, **24**, **25**]. If the ambient 3-manifold has nonnegative curvature, then we can control the topology of these surfaces. Partial results of this type have been proved by various authors [**7**, **16**, **20**, **28**, **29**, **41**]. In this paper we will prove the right sharp estimate for the first time (we remark that the P Schwarz minimal surface of genus 3 has index one in the cubic 3-torus, [**33**]):

Closed index one orientable minimal surfaces immersed in an orientable 3-manifold with nonnegative Ricci curvature have genus ≤ 3.

Finally, we give estimates of the index of a complete minimal surface in term of its topology and we get a sharp estimate for the genus of stable constant mean curvature surfaces in nonnegatively curved 3-manifolds (this family of surfaces includes the solutions of the isoperimetric problem): improving previous results, we obtain that *compact orientable stable constant mean curvature surfaces, immersed in a 3-manifold with nonnegative Ricci curvature, have genus ≤ 3.*

We conclude this introduction with some comments about the strategy of the proofs of our results. If we are considering stability problems on minimal surfaces, or upper bounds for the eigenvalues of Schrödinger operators on surfaces, we need to dispose of simple test functions (or more generally test sections) with controlled energy. Among these we have the ones that rely on the complex or algebraic geometry of the surface. In particular, meromorphic maps (i.e., conformal maps between the surface and the 2-sphere) were used by Yang and Yau [**39**] to obtain an upper bound of the first eigenvalue of the Laplacian on a compact surface in terms of the area and the genus of the surface (generalizing an earlier theorem of Hersch [**13**] for the genus zero case) and then by several authors in other related contexts, see for instance [**7**, **16**, **20**, **28**, **29**, **41**]. Other natural test objects which have been considered in various problems are harmonic forms, see [**23**, **38**], and more generally holomorphic sections in complex bundles, see for instance [**21**, **8**]. In this paper we will use test functions constructed using both meromorphic maps and harmonic forms.

2. Preliminaries

Let Σ be a connected surface immersed in a Riemannian 3-manifold M. If a unit normal vector field N is globally defined on Σ, we say that the immersion is *two-sided*. Otherwise the immersion is said to be *one-sided*. When the ambient manifold M is orientable two-sidedness is equivalent to the orientability of Σ.

If Σ is a two-sided minimal surface, then any compactly supported function $u \in C^1(\Sigma)$ determines a normal deformation of the surface with infinitesimal variation uN. The second variation formula of the area for this deformation is given by

$$(1) \qquad Q(u,u) = \int_\Sigma |\nabla u|^2 - (\mathrm{Ric}\,(N) + |\sigma|^2)u^2,$$

where σ is the second fundamental form of the immersion, $\mathrm{Ric}\,(v)$ is the Ricci curvature of M in the direction of the vector v and N the unit normal vector along Σ. The associated linear operator $\Delta + \mathrm{Ric}\,(N) + |\sigma|^2$, Δ being the Laplacian of Σ, is called the Jacobi operator and satisfies $-\int_\Sigma u(\Delta v + (\mathrm{Ric}\,(N) + |\sigma|^2)v) = Q(u,v)$. Using the Gauss equation we get

$$(2) \qquad \mathrm{Ric}\,(N) + |\sigma|^2 = \mathrm{Ric}\,(e_1) + \mathrm{Ric}\,(e_2) - 2K,$$

where e_1, e_2 is orthonormal basis in the tangent plane of the surface and K is its Gauss curvature. The surface Σ is *stable* if

$$(3) \qquad 0 \le Q(u,u), \text{ for all compactly supported } u \in C^1(\Sigma).$$

If Σ is compact stable and $\mathrm{Ric} \ge 0$, then the immersion is totally geodesic. It follows from Fischer-Colbrie and Schoen [11] that complete two-sided stable minimal surfaces in flat 3-manifolds are planar.

If the minimal surface Σ is one-sided, then the normal bundle is non-trivial and the stability condition is expressed in a different form. It is customary to work on the two-sheeted two-sided covering $\widetilde{\Sigma} \to \Sigma$. If τ denotes the change of sheets involution and N is the unit normal vector on $\widetilde{\Sigma}$, then $N \circ \tau = -N$ and infinitesimal normal deformations of Σ correspond to functions u on $\widetilde{\Sigma}$ satisfying $u \circ \tau = -u$. Therefore, if Q is the second variation formula of the area on $\widetilde{\Sigma}$, then the immersion of Σ is stable if and only if

$$(4) \quad 0 \le Q(u,u), \ \forall u \in C^1(\widetilde{\Sigma}) \text{ with compact support and } u \circ \tau = -u.$$

In the rest of this section we explain some properties of meromorphic maps and harmonic 1-forms defined on a Riemann surface. These objects will be used as test functions to deduce geometric consequences from stability and related hypothesis.

2.1. Harmonic forms. Let Σ be an orientable Riemannian surface and $H^1(\Sigma, \mathbb{R})$ the space of harmonic 1-forms. Recall that a 1-form ω on Σ is harmonic if and only if it is closed, i.e., $(\nabla\omega)(x, y) = (\nabla\omega)(y, x)$ for all tangent vectors x and y ($\nabla\omega$ being the covariant derivative of ω), and coclosed, that is $\nabla\omega(e_1, e_1) + \nabla\omega(e_2, e_2) = 0$ (e_1, e_2 being an orthonormal base in the tangent plane of Σ). Let us denote by K and Δ the Gauss curvature and the rough Laplacian of Σ, respectively. In particular, Δ acts on 1-forms as $\Delta\omega(x) = \nabla^2\omega(e_1, e_1, x) + \nabla^2\omega(e_2, e_2, x)$. It is well-known that if ω is harmonic, then

$$(5) \qquad \Delta\omega - K\omega = 0.$$

Therefore, given $\omega, \omega' \in H^1(\Sigma, \mathbb{R})$, we obtain

$$(6) \qquad \Delta\langle\omega, \omega'\rangle - 2K\langle\omega, \omega'\rangle = 2\langle\nabla\omega, \nabla\omega'\rangle.$$

We also recall that, in the case that Σ is compact, then ω is harmonic if and only if it satisfies equation (5). Moreover, if Σ has genus g, then the space of harmonic 1-forms $H^1(\Sigma, \mathbb{R})$ is $2g$-dimensional.

If Σ is an orientable nonplanar minimal surface immersed in $\mathbb{R}^3$, then the differentials of the linear functions of the immersion dx_1, dx_2, dx_3 are harmonic and generate a 3-dimensional subspace $\mathcal{L}(\Sigma)$ of $H^1(\Sigma, \mathbb{R})$. The conjugate harmonic functions generate another 3-dimensional subspace $\mathcal{L}^*(\Sigma) = \text{span}\{dx_1^*, dx_2^*, dx_3^*\} \subset H^1(\Sigma, \mathbb{R})$. If we denote by $N = (N_1, N_2, N_3)$ and σ the unit normal vector and the second fundamental form of the immersion, respectively, we have that

$$(7) \qquad (\nabla dx_j)(x, y) = \sigma(x, y)N_j \ \text{ and } \ (\nabla dx_j^*)(x, y) = \sigma(Jx, y)N_j,$$

J being the complex structure of Σ and $j = 1, 2, 3$. Given $\omega \in H^1(\Sigma, \mathbb{R})$, we consider the dual tangent vector field (viewed as a $\mathbb{R}^3$-valued function) $X_\omega : \Sigma \to \mathbb{R}^3$, defined by

$$(8) \qquad X_\omega = (\langle\omega, dx_1\rangle, \langle\omega, dx_2\rangle, \langle\omega, dx_3\rangle).$$

We will need the following basic properties of the second fundamental form:

   i) σ is a trace zero symmetric tensor.
  ii) $\nabla\sigma$ is symmetric in all its three arguments (Codazzi equation).
 iii) $\sigma(Jx, y) = \sigma(x, Jy)$, and so, $\sigma^*(x, y) = \sigma(Jx, y)$ is also a trace zero symmetric tensor. Moreover σ and σ^* are (pointwise) orthogonal.
 iv) $\sum_i \sigma(x, e_i)\sigma(e_i, y) = -K\langle x, y\rangle$ and $|\sigma|^2 = -2K$ (Gauss equation).

Lemma 1. *Let Σ be an orientable nonflat minimal surface immersed in $\mathbb{R}^3$ and ω a harmonic 1-form on Σ. Then*

$$(9) \qquad \Delta X_\omega + |\sigma|^2 X_\omega = 2\langle\nabla\omega, \sigma\rangle N.$$

Moreover $\langle\nabla\omega, \sigma\rangle \equiv 0$ if and only if $\omega \in \mathcal{L}^*(\Sigma)$.

Proof. Using (6), (7) and iv), we obtain (9).

As planar points are isolated, to prove the last assertion of the lemma we can assume that $\sigma \neq 0$ at every point of Σ. Clearly, the covariant derivative of any harmonic 1-form is a linear combination of σ and σ^*. Thus, $\langle \nabla \omega, \sigma \rangle \equiv 0$ implies that there is a function h on Σ such that

$$(10) \qquad \nabla \omega(x, y) = h \sigma(Jx, y).$$

By taking covariant derivative with respect to a tangent vector z, we get

$$\nabla^2 \omega(z, x, y) = \nabla h(z) \, \sigma(Jx, y) + h \nabla \sigma(z, Jx, y).$$

From the curvature identity of Ricci, using ii) and iii), we obtain

$$-\omega \left(R(z, x)y \right) = \nabla^2 \omega(z, x, y) - \nabla^2 \omega(x, z, y)$$

$$= \nabla h(z) \, \sigma(Jx, y) - \nabla h(x) \, \sigma(Jz, y),$$

where $R(z, x)y = K(\langle x, y \rangle z - \langle z, y \rangle x)$ is the curvature operator of the surface. The contraction $x = y = e_i$ transforms the above identity into

$$K \omega(z) = \sum_i \nabla h(e_i) \sigma(Jz, e_i),$$

and, if we choose $z = \sum \sigma(Je_k, v)e_k$ from iv), we have

$$K \sum_k \omega(e_k) \sigma(Je_k, v) = \sum_{ik} \nabla h(e_i) \sigma(Je_k, e_i) \sigma(Je_k, v) = -K \nabla h(v).$$

In conclusion, we have shown that the pair (ω, h) is a solution of the first order total linear differential equation

$$(11) \qquad \nabla \omega(x, y) = h \sigma(Jx, y) \qquad \nabla h(x) = -\sigma(JX_\omega, x).$$

Note that, as each solution is determined by its value at a point $p \in \Sigma$, the set of solutions of (11) is a vector space of dimension at most three. On the other hand, for each vector $a \in \mathbb{R}^3$ we have the explicit solution of

$$\omega(x) = \langle Jx, a \rangle \qquad h = \langle N, a \rangle \qquad X_\omega = N \wedge a.$$

Therefore $\{\omega \in H^1(\Sigma, \mathbb{R}) \, | \, \langle \nabla \omega, \sigma \rangle = 0\} = \mathcal{L}^*(\Sigma)$ and the lemma is proved.   q.e.d.

Palmer [**23**] used the tangent vectors fields X_ω to study the index, as a harmonic map, of the Gauss map of a closed constant mean curvature surface in $\mathbb{R}^3$.

2.2. Meromorphic maps. The following construction is very useful in the study of stability and index problems for minimal and constant mean curvature surfaces: let $\phi : \Sigma \to S^2(1)$ be a nonconstant meromorphic map on a compact orientable Riemannian surface (viewed as a $\mathbb{R}^3$-valued map by using a conformal identification between $\overline{\mathbb{C}}$ and the unit 2-sphere). The quadratic form

$$(12) \qquad Q_\phi(u, u) = \int_\Sigma |\nabla u|^2 - |\nabla \phi|^2 u^2, \qquad u \in C^1(\Sigma),$$

does not depend on the metric but only on its conformal class. In particular, the kernel and the index (i.e., the number of negative eigenvalues) of the associated operator $\Delta + |\nabla \phi|^2$ are conformal invariants. Henceforth, when considering these invariants we can work without being precise about the metric in the conformal class we have chosen. The linear coordinates of ϕ lie in the kernel, that is, $\Delta \phi + |\nabla \phi|^2 \phi = 0$. The first eigenvalue of the operator is always negative. So the index one case is the simplest possibility in this context. If φ_1 is the first eigenfunction of $\Delta + |\nabla \phi|^2$ the operator has index one if and only if

$$Q_\phi(u, u) \geq 0 \text{ for any } u \in C^1(\Sigma) \text{ such that } \int_\Sigma u\varphi_1 = 0.$$

Note that φ_1 is positive. The next lemma was first proved by Hersch [13] in case ϕ is the identity map and then adapted to more general situations in [39, 14].

Lemma 2. *Given a nonconstant meromorphic map $\phi : \Sigma \to S^2(1) \subset \mathbb{R}^3$, Σ being a compact Riemannian surface, and a positive continuous function φ on Σ, there exists a Moebius transformation $t : S^2 \longrightarrow S^2$ such that $\int_\Sigma (t \circ \phi)\varphi = 0$.*

Montiel and Ros [20] proved the following result, see also [16].

Lemma 3. *Let $\phi : \Sigma \to S^2(1)$ be a nonconstant meromorphic map from a compact Riemannian surface such that the operator $\Delta + |\nabla \phi|^2$ has index one. Then the degree of any other nonconstant meromorphic map ϕ' is larger that or equal to the degree of ϕ, and the equality holds if and only if there is a Moebius transformation $t : S^2 \longrightarrow S^2$ such that $\phi' = t \circ \phi$.*

This lemma, combined with known results from Algebraic Geometry, allows us to conclude the following theorem; compare with [20, 28].

Theorem 4. *Let Σ be a compact Riemann surface of genus g and $\phi : \Sigma \to S^2(1)$ a nonconstant meromorphic map of degree d. Assume the operator $\Delta + |\nabla \phi|^2$ has index one. Then*

$$d \leq \left[\frac{g}{2}\right] + 1,$$

where $[z]$ denotes the integer part of z.

Proof. The theorem follows directly from Lemma 3 and the properties below. From the Brill-Noether theory, see [**1**], we have that any closed Riemann surface admits a nonconstant meromorphic map of degree $d \leq [\frac{g+1}{2}] + 1$. That proves the theorem when g is even. If g is odd and $d = [\frac{g+1}{2}] + 1$, then the Brill-Noether theorem asserts that the meromorphic map is not unique up to Moebius transformations. So, in the case g is odd we have $d \leq [\frac{g+1}{2}]$.   q.e.d.

Particularizing the theorem to small genus cases we get that, if $\Delta + |\nabla\phi|^2$ has index one, then

$g = 0 \Rightarrow d = 1$,

$g \neq 1$,

$g = 2, 3 \Rightarrow d = 2$,

$g = 4, 5 \Rightarrow d \leq 3$.

For nonorientable minimal surfaces, stability can be also reformulated in terms of meromorphic maps as follow: besides the orientable Riemannian surface Σ and the meromorphic map ϕ, we consider a fixed point free anticonformal involution τ on Σ such that $\phi \circ \tau = -\phi$ (the nonorientable surface here is Σ/τ). The relevant condition to be considered now is

(13) $Q_\phi(u, u) \geq 0$ for all $u \in C^1(\Sigma)$ such that $u \circ \tau = -u$.

The following lemma has been used several times in the study of the stability of nonorientable minimal surfaces. It was obtained by Ross [**34**], who adapted the arguments of [**16**].

Lemma 5. *Let Σ be a compact orientable Riemannian surface, τ an anticonformal involution without fixed points and $\phi : \Sigma \to S^2(1)$ a meromorphic map satisfying $\phi \circ \tau = -\phi$ such that the quadratic form Q_ϕ verifies* (13). *Then any meromorphic map ϕ' on Σ with $\phi' \circ \tau = -\phi'$ satisfies* degree $(\phi') \geq$ degree (ϕ) *and the equality holds if and only if ϕ and ϕ' differ in a Moebius transformation.*

Particularizing to the case *genus* ≤ 1, we conclude

Theorem 6 ([**34**]). *In the hypothesis of the Lemma:*
a) *If* genus $(\Sigma) = 0$, *then the property* (13) *implies* degree $(\phi) = 1$.
b) *If* genus $(\Sigma) = 1$, *then* (13) *does not hold.*

Proof. In the genus zero case we can assume that τ is the antipodal map. So we can take ϕ' as the identity and we get a). To prove b) we use the conformal automorphisms of the torus to find meromorphic maps ϕ_i, $i = 1, 2$, of degree two satisfying $\phi_i \circ \tau = -\phi_i$ and which are not related by a conformal transformation of the sphere.   q.e.d.

3. Stability of one-sided minimal surfaces

If we have a nonorientable Riemannian surface Σ, we can consider the 2-sheeted orientable covering $\widetilde{\Sigma} \to \Sigma$ and the change of sheets involution $\tau : \widetilde{\Sigma} \to \widetilde{\Sigma}$. The space of harmonic 1-forms $H^1(\widetilde{\Sigma}, \mathbb{R})$ decomposes into two subspaces $H^1(\widetilde{\Sigma}, \mathbb{R}) = H^+(\widetilde{\Sigma}) \oplus H^-(\widetilde{\Sigma})$ defined by $H^\pm(\widetilde{\Sigma}) = \{\omega \mid \tau^*(\omega) = \pm\omega\}$, and the Hodge star operator of $\widetilde{\Sigma}$ transforms H^+ into H^-. Hence, when $\widetilde{\Sigma}$ is a compact surface of genus g, then we get that $\dim H^+ = \dim H^- = g$.

Note that, if Σ is minimally immersed in $\mathbb{R}^3$, then $\mathcal{L}(\widetilde{\Sigma}) \subset H^+(\widetilde{\Sigma})$ and $\mathcal{L}^*(\widetilde{\Sigma}) \subset H^-(\widetilde{\Sigma})$.

Theorem 7. *Let Σ be a nonorientable compact stable minimal surface immersed in a flat three torus T^3. Then Σ has the topology of a Klein bottle with a handle.*

Proof. As $\mathcal{L}(\widetilde{\Sigma}) \subset H^+(\widetilde{\Sigma})$ and the surface is nonflat, we deduce that $g \geq 3$ (this fact can be also proved by looking at the Gauss map of $\widetilde{\Sigma}$, see Meeks [**17**]).

If $\omega \in H^-(\widetilde{\Sigma})$ then $X_\omega \circ \tau = -X_\omega$ (in the same way, $\omega \in H^+ \Rightarrow X_\omega \circ \tau = X_\omega$). Hence, we can use (the linear coordinates of) $X = X_\omega$ as a test function in the stability inequality (4), and we conclude that

$$0 \leq \sum_{j=1}^{3} Q(X_j, X_j) = Q(X, X) = -\int_{\widetilde{\Sigma}} \langle \Delta X + |\sigma|^2 X, X \rangle$$

$$= -2 \int_{\widetilde{\Sigma}} \langle \nabla \omega, \sigma \rangle \langle X, N \rangle = 0,$$

where the last equality follows because X is tangent and N is normal to the surface. This implies that X lies in the kernel of Q, that is $\Delta X + |\sigma|^2 X = 0$. From Lemma 1 we deduce that $\omega \in \mathcal{L}^*(\widetilde{\Sigma})$ and, therefore, $g = 3$. So it follows that Σ is a Klein bottle with a handle and that proves the theorem.   q.e.d.

Minimal Klein bottles with a handle immersed in a flat 3-torus $x : \Sigma \to T$ have total curvature -4π and they have been considered by Meeks in [**17**]. He proves that there exists a minimal embedding $x' : \Sigma \to T'$ in some three torus and a covering map $\pi : T' \to T$ such that $x = \pi \circ x'$. Moreover the conjugate minimal surface defines a second isometric minimal embedding of Σ in another 3-torus T''.

The following basic fact will be used in the proof of the main theorem below.

Lemma 8. *Let Σ be a nonorientable surface and $\widetilde{\Sigma} \to \Sigma$ the two-sheeted orientable covering. If $\widetilde{\Sigma}$ has finite genus, then there is a compact set $F \subset \Sigma$ such that $\Sigma - F$ is orientable.*

Proof. Denote by τ the deck transformation of $\widetilde{\Sigma}$ given by change of sheet. As Σ is nonorientable, there exists a Jordan curve γ_1 in Σ such that its pullback image $\widetilde{\gamma}_1 \subset \widetilde{\Sigma}$ is connected and its complement $\widetilde{\Sigma} - \widetilde{\gamma}_1$ has one or two connected components. Moreover, in the second case τ permutes these components and so $\Sigma - \gamma_1$ is orientable. If $\Sigma - \gamma_1$ is nonorientable, then we can choose a second Jordan curve $\gamma_2 \subset \Sigma - \gamma_1$ with connected pullback image $\widetilde{\gamma}_2$ disjoint with $\widetilde{\gamma}_1$, such that either $\widetilde{\Sigma} - (\widetilde{\gamma}_1 \cup \widetilde{\gamma}_2)$ is connected or $\Sigma - (\gamma_1 \cup \gamma_2)$ is orientable. If $\Sigma - (\gamma_1 \cup \gamma_2)$ is nonorientable, we apply the argument again. As $\widetilde{\Sigma}$ has finite genus, after a finite number of times we get pairwise disjoint Jordan curves $\gamma_1, \ldots, \gamma_g$ in Σ such that $\Sigma - (\gamma_1 \cup \cdots \cup \gamma_g)$ is orientable, as we claimed.
$$\text{q.e.d.}$$

Theorem 9. *Let Σ be a complete stable minimal surface in $\mathbb{R}^3$. Then Σ is a plane.*

Proof. In the orientable case the theorem was proved by Do Carmo and Peng [**6**], Fisher-Colbrie and Schoen [**11**] and Pogorelov [**26**], independently. So we assume that Σ is non orientable. In this case, the result has been proved by Ross [**34**], under the aditional assumption that the surface has finite total curvature. Consider the 2-sheeted orientable covering $\widetilde{\Sigma} \to \Sigma$ and the change of sheets involution τ. We discuss two situations separately.

Case 1: genus $(\widetilde{\Sigma}) \geq 2$ (*including the case genus* $= \infty$). From the classic theory of Riemann surfaces, see [**9**] p. 42, we know that $\dim H^1(\widetilde{\Sigma}, \mathbb{R}) \cap L^2(\widetilde{\Sigma}) \geq 4$: in fact, each nonseparating Jordan curve $\gamma \subset \widetilde{\Sigma}$ produces, in a canonical way, a square integrable harmonic 1-form ω_γ which is exact on $\widetilde{\Sigma} - \gamma$ and such that the period of ω_γ along the curve γ' is equal to 1, γ' being a closed curve intersecting γ transversally just at one point. Thus, if we take two disjoint Jordan curves γ and η in $\widetilde{\Sigma}$ such that $\widetilde{\Sigma} - \gamma \cup \eta$ is connected, then the harmonic 1-forms ω_γ, ω_η, ω_γ^* and ω_η^* are linearly independent (here ω^* denotes the image of ω by the Hodge star operator of $\widetilde{\Sigma}$). In particular we deduce that $\dim H^-(\widetilde{\Sigma}) \cap L^2(\widetilde{\Sigma}) \geq 2$.

Take $\omega \in H^-(\widetilde{\Sigma}) \cap L^2(\widetilde{\Sigma})$ and $X = X_\omega$. Take also compactly supported smooth maps $Y : \widetilde{\Sigma} \to \mathbb{R}^3$ and $\varphi : \widetilde{\Sigma} \to \mathbb{R}$ with $Y \circ \tau = -Y$, $\varphi \circ \tau = \varphi$ where φ is a cutoff function satisfying $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on a neighborhood of the support of Y and $|\nabla \varphi| \leq 1$. Using $X_t = \varphi X + tY = \varphi(X + tY)$, $t \in \mathbb{R}$, as test function for the stability condition we have $Q(X_t, X_t) \geq 0$, for each t. As the above is a quadratic expression in t and $Q(X, Y) = Q(\varphi X, Y)$, this is equivalent to say that

$$(14) \qquad\qquad Q(X, Y)^2 \leq Q(Y, Y) Q(\varphi X, \varphi X).$$

From equation (9), after several integration by parts, we obtain

$$
\begin{aligned}
Q(\varphi X, \varphi X) &= \int_{\widetilde{\Sigma}} |\nabla(X\varphi)|^2 - |\sigma|^2 |X\varphi|^2 \\
&= \int_{\widetilde{\Sigma}} -\langle \Delta(X\varphi), X\varphi \rangle - |\sigma|^2 |X\varphi|^2 \\
&= \int_{\widetilde{\Sigma}} -\langle \Delta X + |\sigma|^2 X, X \rangle \varphi^2 - \varphi \Delta\varphi |X|^2 - 2\langle \nabla\varphi X, \varphi\nabla X \rangle \\
&= \int_{\widetilde{\Sigma}} -\varphi\Delta\varphi |X|^2 - \frac{1}{2}\langle \nabla\varphi^2, \nabla|X|^2 \rangle \\
&= \int_{\Sigma} -\varphi\Delta\varphi |X|^2 + \varphi\Delta\varphi |X|^2 + |\nabla\varphi|^2 |X|^2 \\
&= \int_{\widetilde{\Sigma}} |\nabla\varphi|^2 |X|^2.
\end{aligned}
$$

As $|X| \in L^2(\widetilde{\Sigma})$, the last integral converges to zero when we take the region $\{\varphi = 1\}$ larger and larger and so, we obtain $Q(X, Y) = 0$ using (14). Therefore, from (9) we deduce

$$
-2\int_{\widetilde{\Sigma}} \langle \nabla\omega, \sigma \rangle \langle N, Y \rangle = \int_{\widetilde{\Sigma}} -\langle \Delta X + |\sigma|^2 X, Y \rangle = Q(X, Y) = 0
$$

for all anti-invariant Y. Thus $\langle \nabla\omega, \sigma \rangle \equiv 0$ and then $\omega \in \mathcal{L}^*(\widetilde{\Sigma}) \cap L^2(\widetilde{\Sigma})$. This argument gives us two linear functions of the immersion $\widetilde{\Sigma} \to \mathbb{R}^3$, say x_1 and x_2, with finite energy, and so

$$
\int_{\widetilde{\Sigma}} 1 \leq \int_{\widetilde{\Sigma}} |\nabla x_1|^2 + |\nabla x_2|^2 < \infty,
$$

which contradicts the well-known fact that the area of a complete minimal surface in $\mathbb{R}^3$ is infinite, see the final remarks in Yau [40].

Case 2: genus $(\widetilde{\Sigma}) \leq 1$. Now we follow Ross [34], see also [15]: Using Lemma 8 we have that Σ is orientable outside of a compact set. From Fischer-Colbrie [10] we get that Σ has finite total curvature. Therefore $\widetilde{\Sigma}$ is conformally equivalent to a finitely punctured compact Riemann surface $\overline{\Sigma}$ of genus ≤ 1 and the Gauss map extends to a meromorphic map $N : \overline{\Sigma} \to S^2(1)$. Moreover $N \circ \tau = -N$ and the stability condition can be rewritten in the compactified surface (using a metric on $\overline{\Sigma}$ compatible with the conformal structure) as $\int_{\overline{\Sigma}} |\nabla u|^2 - |\nabla N|^2 u^2 \geq 0$ for any anti-invariant function u on $\overline{\Sigma}$. As consequence of Lemma 5 we conclude that $genus(\overline{\Sigma}) = 0$ and $degree(N) = 1$. Then it is well-known, see [22], that the immersion of $\widetilde{\Sigma}$ gives either the Catenoid or the Enneper surface, none of which factorizes through a nonorientable surface. This contradiction finishes the proof of the theorem. q.e.d.

Using Theorem 9, we can extend the curvature estimate of Schoen [**37**] to one-sided stable minimal surfaces.

Corollary 10. *There is a constant $c > 0$ such that any stable minimal surface (either one-sided or two-sided) Σ immersed in $\mathbb{R}^3$ satisfies the curvature estimate*

$$|\sigma| \leq c \, r^{-1},$$

where σ is the second fundamental form of the immersion and r is the intrinsic distance to the boundary of Σ.

Proof. The corollary follows from the uniqueness of the plane among complete stable minimal surfaces in $\mathbb{R}^3$ combined with a blow-up argument, see [**5**], [**18**] and [**31**]. In fact, assuming that the estimate does not hold, we can produce after rescaling and suitable choosing of sub-surfaces, a sequence of minimal surfaces Σ_n immersed in $\mathbb{R}^3$ and points $x_n \in \Sigma_n$ such that, if we denote by $|\sigma_n|$ and r_n the length of second fundamental form and the distance to the boundary on Σ_n, respectively, we have the following:

i) $|\sigma_n|(x_n) = 1$, for each n,
ii) $r_n(x_n)$ goes to infinity, and
iii) for each n, the function $r_n|\sigma_n| : \Sigma_n \to \mathbb{R}$ attains its maximum at x_n.

Passing to the limit we obtain a complete minimal surface immersed in $\mathbb{R}^3$ which is stable and nonflat. This contradicts Theorem 9 and the corollary is proved.   q.e.d.

The above arguments can be adapted to give curvature estimates for stable minimal surfaces in 3-manifolds M with injectivity radius bounded from below and bounded curvatures and derivatives of curvatures. In particular, we have the following result.

Corollary 11. *Let M be a complete 3-manifold with bounded geometry. Then there is a constant $c > 0$, depending on a lower bound of the injectivity radius of M and bounds of the curvature of M and its derivatives, such that any (either one-sided or two-sided) complete stable minimal surface Σ immersed in M satisfies $|\sigma| \leq c$.*

Now we consider the stability condition for one-sided complete minimal surfaces in some quotients of $\mathbb{R}^3$. In the two-sided case, this is the same as the stability problem in the euclidean space itself (it is proved in [**11**] that if a two-sided minimal surface is stable, then its universal covering is stable, too). On the contrary, in the one-sided case different quotient spaces present different kinds of complete stable minimal surfaces.

Theorem 12. *Let $f : \Sigma \to M = \mathbb{R}^3/\Gamma$ be a complete noncompact and nonorientable minimal surface immersed in the quotient of the Euclidean space by a discrete group of translations Γ. If Σ is stable, then it is either (a quotient of) the Helicoid or a (quotient of the) double periodic Scherk's surface. In the first case the surface is a Moebius strip and in the second one a twice punctured projective plane. In both cases the total curvature equals -2π.*

Proof. We keep the notation of the proof of Theorem 9 and we discuss the same cases as there.

The case genus $(\widetilde{\Sigma}) \geq 2$ is shown to be impossible just as in the proof above: although now the linear coordinates are not univalued, the fact we need in the argument is only that the differential of the immersion (dx_1, dx_2, dx_3) is globally well-defined. So we get that Σ has finite area. As Corollary 11 says that Σ has bounded curvature, we also have that $\int_\Sigma |K| < \infty$. These two finiteness conditions are clearly incompatible for complete noncompact minimal surfaces and we conclude that this case is impossible.

In the case genus $(\widetilde{\Sigma}) \leq 1$, arguing as in the proof of Theorem 9 we get that the genus is 0 and the immersion of $\widetilde{\Sigma}$ has total curvature -4π. The Cohn-Vossen inequality implies that $\widetilde{\Sigma}$ has at most four ends. These surfaces can be studied by looking at its Weierstrass representation. They are parameterized by its Gauss map and, because of the existence of the nonorientable quotient, we get that the ends consist of one or two pairs of antipodal points. Under these assumptions we obtain either the Helicoid or the doubly periodic Scherk surface or a surface in the one parameter family of isometric surfaces associated to the two above. Again the existence of the orientation reversing translation excludes all the surfaces but the Helicoid and the doubly periodic Scherk surfaces, itself. In fact, if ψ_d and ψ_s denote the double and the single periodic Scherk surfaces respectively (these surfaces are multivalued and ψ_s is the conjugate of ψ_d), then the anticonformal involution τ induces a translational period on ψ_d and a screw motion on ψ_s. The associated surfaces are given by $\cos\theta\,\psi_d + \sin\theta\,\psi_s$ and so the only way to have a translation period on the associated surface is to take $\sin\theta = 0$. The same argument applies to the Helicoid.   q.e.d.

Let $x_0 : \Sigma \to \mathbb{R}^3/\Gamma_0$ be the standard embedding of the nonorientable doubly periodic Scherk surfaces. Then Γ_0 is an horizontal rectangular planar lattice and the ends are asymptotic to the vertical planes containing the diagonals of the rectangle. The moduli of these surfaces coincides with the one of the rectangles and so it comes in a 1-parameter family. The immersions obtained in the Theorem are of the form $x = \pi \circ x_0$ where π is the linear projection of $\mathbb{R}^3/\Gamma_0$ onto $\mathbb{R}^3/\Gamma$ and $\Gamma_0 \subset \Gamma$. Embeddedness holds only for the case $\Gamma = \Gamma_0$. In the same way the immersions

of the Helicoid consist of the standard imbedding of the nonorientable Helicoid followed by a linear projection.

Now we consider properly embedded minimal surfaces Σ in a complete 3-manifold M which are area minimizing, mod 2. This is the same as saying that each smooth compact subdomain Ω of Σ is area minimizing among compact surfaces (both orientable and nonorientable) with the same boundary. The plane is well known to be the only proper area minimizing surface in $\mathbb{R}^3$. As an interesting consequence of Theorem 12, we obtain a complete classification of global area minimizing surfaces (mod 2) in some flat 3-manifolds.

Theorem 13. *Let Γ be a discrete group of translations of $\mathbb{R}^3$ with* $\mathrm{rank}\,(\Gamma) \leq 2$ *and* $\Sigma \subset \mathbb{R}^3/\Gamma$ *a proper nonplanar area minimizing surface mod 2.*

1) *If Γ is generated by a single translation, then Σ is a nonorientable (quotient of the) Helicoid with total curvature -2π.*
2) *If $\mathrm{rank}\,(\Gamma) = 2$, then Γ is generated by two orthogonal translations and Σ is a nonorientable (quotient of the) doubly periodic Scherk surface with total curvature -2π.*

Proof. From Theorem 12 we only need to show that both the Helicoid and the Scherk surface are area minimizing. Consider the compact flat 3-manifolds $M = \mathbb{R}^3/\mathcal{G}$, where $\mathcal{G}$ is the group generated by two linearly independent horizontal translations $v_1 = (1, 0, 0)$ and $v_2 = (0, 0, a)$ and the glide reflection $(x, y, z) \mapsto (-x, y + b, z)$, with $a, b > 0$ (this is the group Pc in international crystallographic notation). Note that M is the metric product of a Klein bottle and a circle and that $\mathcal{G}$ contains an index two lattice spanned by v_1, v_2 and $(0, 2b, 0)$.

Claim. There is a nonzero homology class $\alpha \in H_2(M, \mathbb{Z}_2)$ which cannot be represented by an embedded planar surface.

Consider the fundamental region of the space M given in Figure 1. The faces defined by the vertices $1, 4, 10, 7$ and $3, 6, 12, 9$ are identified by a translation which preserves the order of the vertices and the same holds for the faces $1, 3, 6, 4$ and $7, 9, 12, 10$. The faces $1, 3, 9, 7$ and $6, 4, 10, 12$ are identified by a glide reflexion with mirror plane $2, 5, 11, 8$ and translation vector the one connecting 2 and 5. The second homology group of M is $H_2(M, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and a basis of it is given by $\alpha_1 = [1, 3, 6, 4]$, $\alpha_2 = [1, 3, 9, 7]$ and $\alpha_3 = [3, 6, 12, 9]$ (note that the first surface is a Klein bottle while the two other ones are tori). Other classes that can be represented by planar surfaces are $\alpha_1 + \alpha_2 = [7, 9, 6, 4]$ and $\alpha_2 + \alpha_3 = [2, 5, 11, 8]$ (these surfaces are a Klein bottle and a torus respectively).
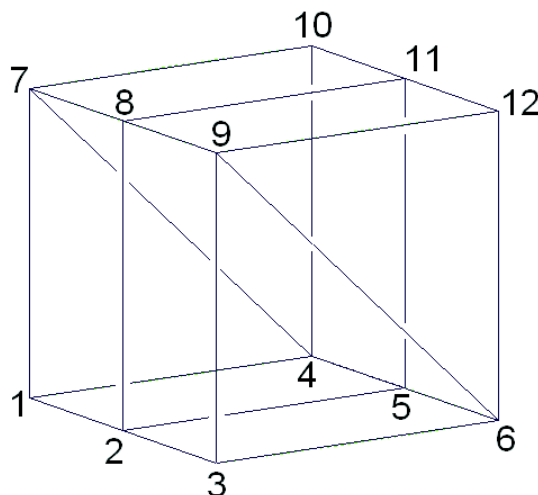
Figure 1. Fundamental region of the Klein bottle times circle.

Each (nonnecessarily connected) embedded planar surface in M is either parallel or perpendicular to $3, 6, 12, 9$. In the first case its homology class lies in $span\{\alpha_2, \alpha_3\}$. In the second one the class of the surface must be linear combination of α_1 and α_2. Therefore the classes $\alpha_1 + \alpha_3$ and $\alpha_1 + \alpha_2 + \alpha_3$ do not contain any planar surface, as we claimed.

Let $\Sigma \subset M$ be an area minimizing surface in the class α. Then Σ is embedded, free of singularities and nonflat (in particular, Σ must be one-sided). Up to a translation of the glide plane, we can assume that Σ passes through the origin and that the maximum of the absolute Gaussian curvature of Σ is attained at that point, $|K|(0) = c^2$. By the maximum principle we can deduce that Σ has an embedded tubular neighborhood in M or any radius $r < 1/c$, see [**30**]. The argument goes as follows: the parallel surface at distance r, $\Sigma(r)$, is a regular immersed surface which defines a two sheeted covering of Σ. Moreover, it has nonnegative mean curvature when we consider the outward pointing unit normal vector (if the principal curvatures of Σ are $(\lambda, -\lambda)$, then the ones of $\Sigma(r)$ are $(\frac{\lambda}{1-r\lambda}, \frac{-\lambda}{1+r\lambda})$). For $r > 0$ small, $\Sigma(r)$ is embedded and is the boundary of the tubular neighborhood $\Omega(r) = \{p \in M/dist(p, \Sigma) \leq r\}$. If there exists the smaller $r < 1/c$ such that $\Sigma(r)$ has a double point, then this surface has a selfcontact at its mean convex side, and the maximum principle would imply that $\Sigma(r)$ is minimal, which is impossible. This proves the assertion.

As a consequence we obtain the estimate $c \geq 1$: to see that, take $p \in \Sigma$ with normal vector v_1. From the existence of the embedded tubular

neighborhood, we get that the segment $\{p + tv_1 \mid -1/c < t < 1/c\}$ is embedded in M, which gives $c \geq 1$. On the other hand, Corollary 11 implies that c is bounded from above independently of the parameters a and b, if we assume $a, b \leq 1$.

Now we take limits when a and b go to infinity. Up to a subsequence, the least area surface Σ converges (with multiplicity one, because of the existence of embedded tubular neighborhood) to a nonflat $\mathbb{Z}_2$-area minimizing surface Σ_∞ in the quotient manifold $\mathbb{R}^3/\langle v_1 \rangle$. From Theorem 12, the only possibility for Σ_∞ is the Helicoid with total curvature -2π.

To prove that the Scherk surfaces are area minimizing, we argue in the same way fixing the parameter a and letting b going to infinity.

<div align="right">q.e.d.</div>

The combination of Theorems 12 and 13 allows us to classify area minimizing surfaces in any natural classes of surfaces (isotopy, homotopy, etc) in $\mathbb{R}^3/\Gamma$, with $\text{rank}(\Gamma) \leq 2$.

4. Index one minimal surfaces.

First we will improve Theorem 4 in the genus four case. This improvement will allow us to give a sharp bound on the topology of two-sided index one minimal surfaces in nonnegatively curved ambient spaces in Theorem 15. Not surprisingly, the proof of Theorem 14 depends on an argument involving minimal surfaces.

Theorem 14. *Let Σ be a compact Riemann surface of genus 4 and $\phi : \Sigma \to S^2(1)$ a nonconstant meromorphic map of degree d. If the operator $\Delta + |\nabla\phi|^2$ has index one, then $d = 2$.*

Proof. According to Theorem 4, it is enough to exclude the case $d = 3$. Note that we cannot do that only by using Lemma 3, as there are closed nonhyperelliptic Riemann surfaces of genus 4 which admit a unique (up to Moebius transformations) meromorphic map of degree 3. Reasoning by contradiction, assume that $\Delta + |\nabla\phi|^2$ has index one and $d = 3$. From Lemma 3 we have that Σ is not hyperelliptic and that any meromorphic map of degree 3 on Σ is related with ϕ by a conformal transformation of the 2-sphere.

Claim. The meromorphic map $\phi : \Sigma \to S^2$ is the Gauss map of an unbranched multivalued conformal minimal immersion x from Σ to $\mathbb{R}^3$ whose differential dx is globally well-defined on Σ.

The proof depends on some special properties of compact Riemann surfaces of genus four which we present briefly; for complete proofs and comments see [**9**, pp. 108-110]. As the space of holomorphic quadratic differentials on Σ has dimension 9, if $\omega_1, \ldots, \omega_4$ is a basis of the space of holomorphic differentials, then the 10 quadratic differentials $\omega_i\omega_j$,

$1 \leq i \leq j \leq 4$, are linearly dependent. So, there are complex numbers a_{ij} such that $\sum_{ij} a_{ij} \omega_i \omega_j = 0$. The rank of this quadratic form is 3 or 4 and by a change of basis this expression transforms into

(15) $\qquad \omega_1^2 + \omega_2^2 + \omega_3^2 = 0 \quad \text{or} \quad \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 = 0.$

If the rank is 4, then Σ admits two degree 3 meromorphic maps which are not related by a Moebius transformation and, so, this case is discarded. Then the rank of the quadratic expression is 3. Moreover, the unique meromorphic map of degree 3 on Σ (up to Moebius transformations) is given by $g = \omega_3/(\omega_1 - \sqrt{-1}\,\omega_2) : \Sigma \to \overline{\mathbb{C}}$.

If the assertion in the claim holds for a meromorphic map ϕ, then it also holds for any meromorphic map related with ϕ by a Moebius transformation, see for instance [20, 22]. So it is enough to prove the claim for the meromorphic map g. If we consider $\omega = \omega_1 - \sqrt{-1}\,\omega_2$, we can write

$$\omega_1 = \frac{1}{2}(1 - g^2)\,\omega, \qquad \omega_2 = \frac{\sqrt{-1}}{2}(1 + g^2)\,\omega, \qquad \omega_3 = g\,\omega.$$

Note that if g has a pole of order m at a point, then the order of the zero of ω at that point is at least $2m$. As g has three poles and any holomorphic differential on Σ has exactly 6 zeros, it follows that the zeros of ω coincide with the poles of g and that the order of the zero is twice the order of the pole at these points. Therefore, g and ω are the Weierstrass data of a multivalued minimal immersion x from Σ to $\mathbb{R}^3$, the differential of x being the real part of ω_1, ω_2 and ω_3, see [22]. That proves the claim.

From now on we work with the metric induced by x. In particular $|\nabla\phi|^2 = |\sigma|^2$, σ being the second fundamental form of x, and the second variation formula of the area Q coincides with quadratic form Q_ϕ associated to $\Delta + |\nabla\phi|^2$. The differential (dx_1, dx_2, dx_3) of the immersion x is globally well-defined on Σ, which in turn implies that the map $\omega \in H^1(\Sigma, \mathbb{R}) \mapsto X_\omega : \Sigma \to \mathbb{R}^3$ introduced in (8) is well-defined, too. The dimension of the space $V = \{\omega \in H^1(\Sigma, \mathbb{R})| \int_\Sigma \varphi_1 X_\omega = 0\}$, φ_1 being the first eigenfunction of $\Delta + |\sigma|^2$, is at least $2 \times 4 - 3 = 5$. As in the proof of Theorem 7 we get that each $\omega \in V$ satisfies $Q(X_\omega, X_\omega) = 0$, which gives $\Delta X_\omega + |\sigma|^2 X_\omega = 0$, and, thus, $\omega \in \mathcal{L}^*(\Sigma)$. So $V \subset \mathcal{L}^*(\Sigma)$ and this contradiction shows that in the case $genus = 4$ and $d = 3$ we cannot have index one. $\hfill$ q.e.d.

Now we consider the case where Σ is a closed two-sided minimal surface whose second variation formula quadratic form has index one. If φ_1 is the first eigenfunction of $\Delta + |\sigma|^2$, then the index one condition can be written as

(16) $\qquad 0 \leq Q(u, u), \text{ for all } u \in C^1(\Sigma) \text{ such that } \int_\Sigma u\varphi_1 = 0.$

When Σ is a compact two-sided minimal surface immersed in a 3-manifold with nonnegative Ricci curvature, then one can use meromorphic functions as test functions, combined with Theorem 3, to control the topology of Σ; see [**41**, **7**, **20**, **29**, **28**]. However, the bounds obtained in these works are not sharp. As there are genus 3 index one compact minimal surfaces in some flat 3-tori, see [**33**], the result below gives the right estimate.

Theorem 15. *Let Σ be a compact orientable two-sided minimal surface immersed in a 3-manifold M with nonnegative Ricci curvature* $\mathrm{Ric} \geq 0$. *If Σ has index one, then* $\mathrm{genus}\,(\Sigma) \leq 3$.

Proof. Suppose Σ is a connected surface of genus g. From the Brill-Noether theorem, see [**1**], we know there is a nonconstant meromorphic map, $\phi : \Sigma \to S^2(1)$ of degree smaller than or equal to $\leq [(g+1)/2]+1$, where $[z]$ denotes the integer part of z. Using Lemma 2 we can assume that the $\mathbb{R}^3$-valued function ϕ is orthogonal to the first eigenfunction φ_1 of the Jacobi operator of Σ (which can be assumed to be positive). Therefore, we can use the linear components of ϕ as test functions in the stability condition (3). Adding these inequalities and taking into account (2), we get

$$0 \leq Q(\phi, \phi) = \int_\Sigma |\nabla \phi|^2 - (\mathrm{Ric}\,(e_1) + \mathrm{Ric}\,(e_2) - 2K)|\phi|^2.$$

As ϕ is conformal, the first term transforms into $8\pi \, \mathrm{degree}\,(\phi)$. Using that ϕ is of length one and the Gauss-Bonnet theorem, we obtain

$$0 \leq 8\pi([(g+1)/2]+1) + 4\pi(2-2g) - \int_\Sigma \mathrm{Ric}\,(e_1) + \mathrm{Ric}\,(e_2).$$

From this inequality, we conclude that $g \leq 5$. Moreover, in the cases $g = 4$ or 5 the equality holds. This implies that $\mathrm{Ric}\,(e_1) + \mathrm{Ric}\,(e_2) = 0$, the Jacobi operator is $\Delta - 2K$ and ϕ lies in its kernel, $\Delta\phi - 2K\phi = 0$. As any meromorphic map is harmonic, we also have $\Delta\phi + |\nabla\phi|^2\phi = 0$. Therefore $|\nabla\phi|^2 = -2K$ and so the operator $\Delta + |\nabla\phi|^2$ has index one. Moreover, the degree of ϕ is $d = 3$ (in the case $g = 4$) or $d = 4$ (for $g = 5$). As this contradicts Theorem 4 and Theorem 14, we conclude that $g \leq 3$.   q.e.d.

In the case Σ is complete and not compact in a manifold with nonnegative Ricci curvature, Ritoré [**28**] proved that if Σ has index one (i.e., each relatively compact domain in Σ has index one), then $g \leq 3$ and $r \leq 4 - g$, g and r being the genus and the number of ends of Σ, respectively. If Σ is compact, then he shows that $g \leq 4$.

5. Estimates of the index of minimal surfaces

The arguments of the above sections can be easily adapted to obtain estimates of the index of complete minimal surfaces in $\mathbb{R}^3$ or in quotients of $\mathbb{R}^3$. In this section we present briefly the results which can be obtained in this way. See [**3**, **12**, **20**] for other results about the index of minimal surfaces.

Theorem 16. *Let Σ be a nonflat compact minimal surface in a flat 3-torus T^3. Then*

$$\text{index}\,(\Sigma) \geq \frac{2g-3}{3}\,, \quad \text{if } \Sigma \text{ is orientable, and}$$

$$\text{index}\,(\Sigma) \geq \frac{g-3}{3}\,, \quad \text{if } \Sigma \text{ is nonorientable,}$$

where g is either the genus of Σ, when Σ is orientable, or the genus of the orientable two-seeted covering, when Σ is nonorientable.

Proof. Suppose first that Σ is orientable. Denote the index and the genus of Σ by n and g, respectively, and consider the eigenfunctions of the Jacobi operator $\varphi_1, \ldots, \varphi_n$ with negative eigenvalues. The dimension of the space

$$V = \{X_\omega \mid \int_\Sigma X_\omega \varphi_k = 0, k = 1, \ldots n, \ \omega \in H^1(\Sigma, \mathbb{R}^3)\}$$

is at least $2g - 3n$. If $X_\omega \in V$, then the index assumption gives $Q(X_\omega, X_\omega) \geq 0$. On the other hand, we know that in fact the equality holds in this inequality and we can deduce as in the proof of Theorem 7, that $\omega \in \mathcal{L}^*(\Sigma)$ and so $\dim V \leq 3$. Joining the two estimates we have for the dimension of V, we conclude that $2g - 3n \leq 3$. That proves the first assertion of the theorem.

If Σ is nonorientable, we denote its two-seeted orientable covering by $\widetilde{\Sigma}$. If the index of Σ is n, then there are eigenfuntions $\varphi_1, \ldots, \varphi_n$ of the Jacobi operator of $\widetilde{\Sigma}$ with $\varphi_k \circ \tau = -\varphi_k$ (τ being the change of sheet involution) and negative eigenvalues. The argument used in the orientable case applies if we consider test functions in the space

$$V = \{X_\omega \mid \int_\Sigma X_\omega \varphi_k = 0, k = 1, \ldots n, \ \omega \in H^-(\widetilde{\Sigma}, \mathbb{R}^3)\}$$

whose dimension is at least $g - 3n$, g being the genus of $\widetilde{\Sigma}$, and we conclude as above that $g - 3n \leq 3$. q.e.d.

Theorem 17. *Let Σ be a complete nonflat minimal surface in $\mathbb{R}^3$. Then Σ has finite index if and only if it has finite total curvature. Moreover*

$$\text{index}\,(\Sigma) \geq \frac{2g}{3}\,, \quad \text{if } \Sigma \text{ is orientable and}$$

$$\text{index}\,(\Sigma) \geq \frac{g}{3}\,, \quad \text{if } \Sigma \text{ is nonorientable.}$$

*In the orientable case g is the genus of Σ and in the nonorientable one
g is the genus of the two-sheeted orientable covering of Σ.*

Proof. Suppose first that Σ is orientable. Fisher-Colbrie [**10**] proved
that index $(\Sigma) < \infty$ if and only if $\int_\Sigma |\sigma|^2 < \infty$. If the index of Σ is
n, then there are eigenfunctions $\varphi_1, \ldots, \varphi_n \in L^2(\Sigma)$ of $\Delta + |\sigma|^2$ with
negative eigenvalues and such that any compactly supported smooth
function φ L^2-orthogonal to $\varphi_1, \ldots, \varphi_n$ satisfies $Q(\varphi, \varphi) \geq 0$, see Propo-
sition 2 in [**10**]. If the index estimate in the statement of the theorem
does not hold, then there is $X = X_\omega \neq 0$, with $\omega \in L^2(\Sigma) \cap H^1(\Sigma, \mathbb{R})$
and $\int_\Sigma X\varphi_k = 0$, for $k = 0, \ldots n$. After that property we can deduce,
as in the proof of Theorem 9, that ω lies in $\mathcal{L}^*(\Sigma)$: In fact, we have
that $Q(X_t, X_t) \geq 0$ for the $\mathbb{R}^3$-valued compactly supported function
$X_t = \varphi(X + tY + v_1\varphi_1 + \cdots + v_n\varphi_n)$, where t is a real number, φ
is a cut-off function, $Y : \Sigma \to \mathbb{R}^3$ has compact support and satisfies
$\int_\Sigma Y\varphi_k = 0$, $k = 0, \ldots, n$, and the vectors $v_k = v_k(t, Y, \varphi) \in \mathbb{R}^3$ are
choosen such that X_t is L^2-orthogonal to the eigenfunctions $\varphi_1, \ldots, \varphi_n$.
Now we follow, step by step, the proof of Theorem 9 and we conclude
that $\omega \in L^2(\Sigma) \cap \mathcal{L}^*(\Sigma)$. As Σ has finite total curvature, this means
that one of the linear coordinates of Σ is constant. This contradiction
proves the theorem in the orientable case.

In the case Σ is nonorientable, if follows from [**10**] that finite total
curvature implies finite index. To prove the converse result, using again
[**10**], it is enough to prove that finite index implies that the orientable
covering has finite genus. Now it follows that both the finiteness of the
genus and the bound of the g in terms of the index can be obtained by
the same kind of arguments used in the orientable case adapted to the
nonorientable context (as in the proof of Theorem 9). We remark that,
in particular, Proposition 2 in [**10**] is true in the nonorientable case.
q.e.d.

Theorem 18. *Let Σ be a complete noncompact minimal surface in
$\mathbb{R}^3/\Gamma$, Γ being a discrete subgroup of translation of $\mathbb{R}^3$. Then Σ has
finite index if and only if it has finite total curvature and, in this case,
we have*

$$\text{index}(\Sigma) \geq \frac{2g-1}{3}, \quad \textit{if } \Sigma \textit{ is orientable, and}$$

$$\text{index}(\Sigma) \geq \frac{g-1}{3}, \quad \textit{if } \Sigma \textit{ is nonorientable,}$$

g being defined as in Theorem 17.

Proof. The only difference with the proof of Theorem 17 above is that
now we can have $\dim \mathcal{L}^*(\Sigma) \cap L^2(\Sigma) = 1$ (for instance, the Riemann
minimal examples are singly periodic minimal surfaces satisfying this
property). The proof of Theorem 12 gives that this dimension cannot
be larger than 1.   q.e.d.

6. Stability of constant mean curvature surfaces

The understanding of stable constant mean curvature surfaces is one of the key steps in the study of the isoperimetric problem, see [**32**]. Let Σ be a compact (two-sided) surface with constant mean curvature H immersed in a Riemannian 3-manifold M. The surface Σ is said to be *stable (for the isoperimetric problem)* if the second variation formula of the area is nonnegative for any volume preserving deformation of Σ. This is the same as saying that $Q(u, u) \geq 0$ for all smooth functions $u \in C^1(\Sigma)$ with $\int_\Sigma u = 0$, where Q is defined as in (1), see Barbosa and Do Carmo [**2**],

$$Q(u, u) = \int_\Sigma |\nabla u|^2 - (Ric(N) + |\sigma|^2)u^2.$$

The mean value zero condition is the infinitesimal corresponding of the volume constraint. When Σ is a minimal surface, we can use two different notions of stability. When necessary to avoid ambiguity, we speak of *stability for the Plateau problem* or *stability for the isoperimetric problem*. If Ric ≥ 0 and Σ is stable for the isoperimetric problem and nonconnected, then Σ is totally geodesic. Note also that, opposite to the minimal case, the two-sided assumption is the natural one in the context of constant mean curvature surfaces. This justifies the hypothesis of the theorem below.

Theorem 19. *Let Σ be a compact connected orientable two-sided constant mean curvature surface, immersed in a 3-manifold M with nonnegative Ricci curvature. If Σ is stable (for the isoperimetric problem), then genus $(\Sigma) \leq 3$.*

Proof. The first part of the proof is parallel to the one of Theorem 15: Instead of (2), in the constant mean curvature case the Gauss equation gives Ric $(N) + |\sigma|^2 =$ Ric $(e_1) +$ Ric $(e_2) + 4H^2 - 2K$. Using meromorphic maps ϕ of small degree and mean value zero as test functions in the stability quadratic form Q, we obtain that the genus of Σ is smaller than or equal to 5. Moreover, if the genus is 4 or 5, then Σ is a minimal surface of index one (as minimal surface). This contradicts Theorem 15 and, so, the theorem is proved.   q.e.d.

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